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Algebraic properties of BRST coupled doublets

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ABSTRACT: We characterize the dependence on doublets of the cohomology of an arbitrary nilpotent differential s (including BRST differentials and classical linearized Slavnov-Taylor (ST) operators) in terms of the cohomology of the doublets-independent component of s. All cohomologies are computed in the space of integrated local formal power series. We drop the usual assumption that the counting operator for the doublets commutes with s (decoupled doublets) and discuss the general case where the counting operator does not commute with s (coupled doublets). The techniques used are purely algebraic and do not rely on power-counting arguments. The main result is that the full cohomology that includes the doublets can be obtained directly from the cohomology of the doublets-independent component of s. This turns out to be a very useful property in many problems in Algebraic Renormalization.

KEYWORDS: BRST Quantization, BRST Symmetry.

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1. Introduction

The use of BRST doublets is by far a well-established technique in Algebraic Renormalization [1, 2]. In gauge theories the dependence of Green functions of gauge-invariant operators on the gauge parameter [3, 4, 5, 6] can be analyzed in an algebraic way by allowing the gauge parameter to vary as a BRST doublet together with its anticommuting partner [7]. Doublets are also a useful tool in proving the independence of gauge-invariant observables of the background gauge field [8, 9, 10, 11].

A couple of variables (z, w) is said a doublet under the nilpotent differential s if

$$sz = w \, , \qquad sw = 0 \, . \tag{1.1}$$

In the physically relevant cases s is usually identified with the classical BRST differential or the linearized classical Slavnov-Taylor (ST) operator $\mathcal{S}_0$ [1, 2, 12, 13].

When the counting operator for the variables (z, w) commutes with s it is an easy task to show that the cohomology of s is doublets-independent. In the general case the problem is more intricated and deserves a careful algebraic investigation.

In this paper we provide a comprehensive discussion of the dependence of the cohomology of s on doublets (also known as contractible pairs [2]) in the space of integrated local formal power series. We first review the standard result referring to the case of decoupled doublets, where the counting operator for the set of doublets (z_k, w_k)

$$\mathcal{N} \equiv \int d^4x \, \sum_k \left(z_k \frac{\delta}{\delta z_k} + w_k \frac{\delta}{\delta w_k} \right) \tag{1.2}$$

commutes with the differential s. In this case it can be shown [1, 2] that the contribution of the doublets (z_k, w_k) to the cohomology of s is always trivial: if $\mathcal{I}[z, w, \varphi]$ is any s-invariant integrated local formal power series in its arguments and their derivatives, depending on the set of doublets $z = \{z_k\}$, $w = \{w_k\}$ and on a set of other fields and external sources collectively denoted by φ, then there exists an integrated local formal power series $\mathcal{G}[z,w,\varphi]$ such that

$$\mathcal{I}[z,w,\varphi] - \mathcal{I}[0,0,\varphi] = s\mathcal{G}[z,w,\varphi]\,. \tag{1.3}$$

From the point of view of Renormalization theory, eq. (1.3) implies that the anomalies of a model, to be identified with the non-trivial cohomology classes of the classical linearized ST operator $\mathcal{S}_0$ in the sector with Faddeev-Popov (FP) charge +1, are not affected by the introduction of a set of decoupled doublets.

In Algebraic Renormalization decoupled doublets play for instance an important rôle in the cohomological analysis of gauge theories, in order to prove the independence of the cohomology of the classical linearized ST operator $\mathcal{S}_0$ of antighosts and auxiliary fields [1, 2], as well as in the off-shell formulation of the Equivalence Theorem [14].

We wish to point out that eq. (1.3) holds regardless the FP charge of $\mathcal{I}$: it may also happen that $\mathcal{I}$ is a sum of terms with different FP charges, although this does not happen in Renormalization theory.

We then relax the assumption that s and $\mathcal{N}$ commute and discuss the more general case of coupled doublets, for which

$$[s,\mathcal{N}] \neq 0\,. \tag{1.4}$$

Under suitable assumptions, in some cases [15, 16] it is known that one can reduce the problem of handling coupled doublets to the decoupled case by means of a properly chosen change of coordinates in the jet space where s acts.

It is the purpose of this paper to give a full characterization of the dependence of the cohomology of s on coupled doublets.

In order to discuss the problem on general grounds it is convenient to decompose the nilpotent differential s according to the degree induced by $\mathcal{N}$:

$$s = \sum_{j=0}^{\infty} s^{(j)}\,, \tag{1.5}$$

where $s^{(j)}$ is the component of s of degree j. Moreover we split the zero-th order operator $s^{(0)}$ as

$$s^{(0)} = \bar{s}^{(0)} + \int d^4x\, w_k \frac{\delta}{\delta z_k}\,, \tag{1.6}$$

where $\bar{s}^{(0)}$ is (z,w)-independent and only acts on the variables φ. From the nilpotency of s we get that $s^{(0)}$ is nilpotent. This implies together with eq. (1.6) that $\bar{s}^{(0)}$ is also nilpotent.

We will prove along the lines of homological perturbation theory [17, 18, 19] that the cohomology of the nilpotent differential s in the space of integrated local formal power series in $\{\varphi, z, w\}$ and their derivatives is isomorphic to the cohomology of $\bar{s}^{(0)}$ in the space of integrated local formal power series which depend only on φ and their derivatives.

To this extent this result establishes the independence of the cohomology of s of (generally coupled) doublets.

In practical applications independence of the cohomology of doublets proves to be an important tool in the computation of cohomologies, since it allows to restrict the cohomological problem to a smaller space where all variables entering as doublets have been

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discarded. This property has been extensively used in the literature whenever dealing with decoupled doublets [1, 2]. Theorem 1 allows to extend the range of applicability of such a technique to the wider class of coupled doublets.

We stress that we do not rely on power-counting arguments. This in turn allows to apply the results of the present paper to generally non power-counting renormalizable theories, as for instance in the BRST approach to the Equivalence Theorem discussed in [20] or in the on-shell formulation of the Equivalence Theorem [14], where independence of the cohomology of the linearized classical ST operator of coupled doublets guarantees that the relevant ST identities are anomaly-free and can thus be directly imposed order by order in the loop expansion [21, 22].

The results of the paper are valid in the case where z and w are all independent elementary variables. More general situations have been discussed in [2, 15, 16] where this requirement is not met. In this case it happens that the relevant cohomologies can actually depend on the doublets [15, 16].

The paper is organized as follows. The algebraic properties of decoupled doublets are briefly reviewed in section 2. The main results of the paper on coupled doublets are provided in section 3. Finally conclusions are reported in section 4.

2. Decoupled doublets

In this section we review the standard result [1, 2] showing that the contribution of doublets to the cohomology of the differential s is trivial if

$$[s, \mathcal{N}] = 0\,. \tag{2.1}$$

The counting operator $\mathcal{N}$ has been defined in eq. (1.2).

The differential $s : \Sigma \to \Sigma$ is assumed to act on the space Σ of local integrated formal power series depending on the set of doublets (z, w), on other variables collectively denoted by φ and on their derivatives. Σ is a linear space generated by all possible integrated linearly independent local monomials in $\{z, w, \varphi\}$ and their derivatives, with no restrictions on the dimension of the generators.

Any integrated local formal power series $\mathcal{G} \in \Sigma$ can be decomposed as

$$\mathcal{G} = \sum_i \int d^4x\, c_i \mathcal{M}_i(x)\,, \tag{2.2}$$

where $\{\mathcal{M}_i(x)\}$ is a basis of local linearly independent monomials in $\{z, w, \varphi\}$ and their derivatives and c_i are c-number coefficients. $\{\mathcal{M}_i(x)\}$ includes monomials of arbitrarily high dimension.

The differential s acting on Σ can be expressed as

$$s = \int d^4x\, s\varphi(x) \frac{\delta}{\delta\varphi(x)} + \int d^4x\, w(x) \frac{\delta}{\delta z(x)}\,. \tag{2.3}$$

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$s\varphi(x)$ is the s-variation of $\varphi(x)$ and is assumed to be a local formal power series in $\{\varphi, z, w\}$ and their derivatives, whose decomposition on the basis $\{\mathcal{M}_i(x)\}$ is given by

$$s\varphi(x) = \sum_i d_i \mathcal{M}_i(x) \, , \tag{2.4}$$

where d_i are c-number coefficients.

All functional derivatives are assumed to act from the left. We also require that s is nilpotent: $s^2 = 0$.

In practical applications s is usually identified with the BRST differential or the classical linearized ST operator $\mathcal{S}_0$ [1, 2].

The space Σ can be graded according to the counting operator in eq. (1.2):

$$\Sigma = \Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)} + \cdots \, , \tag{2.5}$$

where $\Sigma^{(j)}$ is the eigenspace of eigenvalue j for $\mathcal{N}$:

$$\mathcal{G} \in \Sigma^{(j)} \Rightarrow \mathcal{N}\mathcal{G} = j\mathcal{G} \, . \tag{2.6}$$

The action of the differential s is compatible with the grading induced by $\mathcal{N}$ if

$$[s, \mathcal{N}] = 0 \, . \tag{2.7}$$

If eq. (2.7) is verified, we say that we are dealing with "decoupled doublets". This definition is motivated by the observation that if eq. (2.7) holds true, then no doublet (z_k, w_k) can appear in the s-transformation of any other field.

We follow [23] and introduce the operator

$$\mathcal{K} = \int_0^1 dt \sum_i z_i \lambda_t \frac{\delta}{\delta w_i} \, , \tag{2.8}$$

where the operator λ_t acts as follows

$$\lambda_t X[z, w, \varphi] = X[tz, tw, \varphi] \, . \tag{2.9}$$

In the previous equation φ denotes any set of fields and external sources other than (z, w) on which the integrated local formal power series X might depend. By explicit computation it can be verified that $\mathcal{K}$ is a contracting homotopy for s, since it fulfills the following equation

$$\{s, \mathcal{K}\} X = \iota X \, . \tag{2.10}$$

ι is the projector on the orthogonal complement to the kernel of $\mathcal{N}$:

$$\iota = 1|_{\Sigma^{(0)}\perp} \oplus 0|_{\Sigma^{(0)}} \, , \tag{2.11}$$

where $\Sigma^{(0)}$ is the kernel of the counting operator $\mathcal{N}$ in eq. (1.2) and $\Sigma^{(0)\perp}$ its orthogonal complement.

Assume now that $s\mathcal{I} = 0$. $\mathcal{I}$ can depend on (z, w) and φ. We apply eq. (2.10) to $\mathcal{I}$ and obtain

$$\{s, \mathcal{K}\}\mathcal{I}[z, w, \varphi] = s(\mathcal{K}\mathcal{I}) = \iota\mathcal{I}[z, w, \varphi] = \mathcal{I}[z, w, \varphi] - \mathcal{I}[0, 0, \varphi] \, . \tag{2.12}$$

In the previous equation we have used the fact that $\mathcal{I}[z, w, \varphi]$ is s-invariant. From eq. (2.12) we conclude that the (z, w)-dependent part $\mathcal{I}[z, w, \varphi] - \mathcal{I}[0, 0, \varphi]$ of $\mathcal{I}$ is a s-exact term.

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3. Coupled doublets

Let us now move to the "coupled" case where

$$[s,\mathcal{N}] \neq 0\,. \tag{3.1}$$

In this case the operator $\mathcal{K}$ defined in eq. (2.8) is no more a contracting homotopy for s.

We denote by $\varphi = \{\varphi_i\}$ all the variables different from (z,w) on which s may act. We also adopt a compact notation and assume that the integration over d^4x is implicit in the sum over repeated indices, i.e. we write

$$\varphi_i \frac{\delta X}{\delta \varphi_i} \equiv \int d^4x \sum_i \varphi_i(x) \frac{\delta X}{\delta \varphi_i(x)}\,. \tag{3.2}$$

Then we get the following decomposition for s:

$$s = g_i \frac{\delta}{\delta \varphi_i} + w_i \frac{\delta}{\delta z_i}\,. \tag{3.3}$$

$g_i[\varphi, z, w]$ is the s-variation of φ_i.

We can decompose s according to the degree induced by $\mathcal{N}$:

$$s = \sum_{j=0}^{\infty} s^{(j)}\,, \tag{3.4}$$

where $s^{(j)}$ is the component of s of degree j. By comparison with eq. (3.3) we have explicitly

$$\begin{aligned} s^{(0)} &= g_i^{(0)} \frac{\delta}{\delta \varphi_i} + w_i \frac{\delta}{\delta z_i} \\ s^{(j)} &= g_i^{(j)} \frac{\delta}{\delta \varphi_i}\,, \qquad j = 1, 2, \ldots \end{aligned} \tag{3.5}$$

In the above equation $g_i^{(j)}$ denotes the component of g_i in the eigenspace of eigenvalue j for the counting operator $\mathcal{N}$:

$$\mathcal{N} g_i^{(j)} = j g_i^{(j)}\,. \tag{3.6}$$

$g_i^{(0)}$ is the (z,w)-independent component of g_i. By comparison with eq. (1.6) we see that

$$\bar{s}^{(0)} = g_i^{(0)} \frac{\delta}{\delta \varphi_i}\,. \tag{3.7}$$

In most cases s is to be identified with the classical linearized ST operator $\mathcal{S}_0$. We denote by $S[\phi_i, \phi_i^*, z, w]$ the classical action from which we define the symplectic gradient

$$(S,\cdot) \equiv \frac{\delta S}{\delta \phi_i} \frac{\delta}{\delta \phi_i^*} + \frac{\delta S}{\delta \phi_i^*} \frac{\delta}{\delta \phi_i}\,. \tag{3.8}$$

$S[\phi_i, \phi_i^*, z, w]$ depends on both the fields ϕ_i and the associated antifields ϕ_i^* as well as on the doublets (z_k, w_k). The full classical linearized ST operator is then

$$\mathcal{S}_0 = (S,\cdot) + w_k \frac{\delta}{\delta z_k} = \frac{\delta S}{\delta \phi_i} \frac{\delta}{\delta \phi_i^*} + \frac{\delta S}{\delta \phi_i^*} \frac{\delta}{\delta \phi_i} + w_k \frac{\delta}{\delta z_k}\,. \tag{3.9}$$

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Nilpotency of $\mathcal{S}_0$ follows from the ST identity for S [1]:

$$\frac{1}{2}(S,S) + w_k \frac{\delta S}{\delta z_k} = 0\,. \tag{3.10}$$

The symplectic structure in eq. (3.9) is not essential to prove the results of the present section. In particular, we will only rely on eq. (3.3), which is more general than eq. (3.9).

However, the results presented in this section can be derived in an effective geometrical way [24] if the nilpotent differential s is given by the classical linearized ST operator $\mathcal{S}_0$ in eq. (3.9).

We will prove in theorem 1 that the cohomology of the nilpotent differential s in the space of integrated local formal power series in $\{\varphi, z, w\}$ and their derivatives is isomorphic to the cohomology of $\bar{s}^{(0)}$ in the space of integrated local formal power series which depend only on φ and their derivatives.

We first show that if $\mathcal{I}[z,w,\varphi]$ is a s-closed integrated local formal power series such that $s\mathcal{I}[0,0,\varphi]=0$ then its (z,w)-dependent part $\mathcal{I}[z,w,\varphi]-\mathcal{I}[0,0,\varphi]$ is s-exact.

Lemma 1 *Let $\mathcal{I}[z,w,\varphi]$ be an integrated local formal power series closed under the nilpotent differential s, i.e. it fulfills the Wess-Zumino consistency condition*

$$s\mathcal{I} = 0\,. \tag{3.11}$$

Moreover let us assume that $s\mathcal{I}[0,0,\varphi]=0$. Then we have

$$\mathcal{I}[z,w,\varphi] - \mathcal{I}[0,0,\varphi] = s\mathcal{G}[z,w,\varphi] \tag{3.12}$$

for some integrated local formal power series $\mathcal{G}[z,w,\varphi]$.

Proof. The condition in eq. (3.11) implies

$$w_i \frac{\delta \mathcal{I}}{\delta z_i} = -g_i \frac{\delta \mathcal{I}}{\delta \varphi_i}\,. \tag{3.13}$$

Differentiating eq. (3.13) with respect to w_k we get

$$\frac{\delta \mathcal{I}}{\delta z_k} = (-1)^{\epsilon(w_k)+1} s\left(\frac{\delta \mathcal{I}}{\delta w_k}\right) - \frac{\delta g_i}{\delta w_k}\frac{\delta \mathcal{I}}{\delta \varphi_i}\,, \tag{3.14}$$

so that eq. (3.14) becomes (z_k and w_k have opposite statistics)

$$\frac{\delta \mathcal{I}}{\delta z_k} = (-1)^{\epsilon(z_k)} s\left(\frac{\delta \mathcal{I}}{\delta w_k}\right) - \frac{\delta g_i}{\delta w_k}\frac{\delta \mathcal{I}}{\delta \varphi_i}\,. \tag{3.15}$$

In the previous equations we have denoted by $\epsilon(X)$ the Grassmann parity of X ($\epsilon(X)=0$ if X is bosonic, $\epsilon(X)=1$ if X is fermionic).

We apply to both sides of eq. (3.15) the operator $\int_0^1 dt\, z_k \lambda_t$, where the action of λ_t on the integrated local formal power series X is defined by eq. (2.9).

We get:

$$\int_0^1 dt\, z_k \lambda_t \frac{\delta \mathcal{I}}{\delta z_k} = \int_0^1 dt\, \left((-1)^{\epsilon(z_k)} z_k \lambda_t s\left(\frac{\delta \mathcal{I}}{\delta w_k}\right) - z_k \lambda_t \frac{\delta g_i}{\delta w_k}\frac{\delta \mathcal{I}}{\delta \varphi_i}\right). \tag{3.16}$$

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On the other hand

$$\int_0^1 dt\,(-1)^{\epsilon(z_k)} z_k\lambda_t s\left(\frac{\delta\mathcal{I}}{\delta w_k}\right) = \int_0^1 dt\left((-1)^{\epsilon(z_k)} z_k s\left(\lambda_t\frac{\delta\mathcal{I}}{\delta w_k}\right) - (-1)^{\epsilon(z_k)} z_k[s,\lambda_t]\frac{\delta\mathcal{I}}{\delta w_k}\right)$$
$$= \int_0^1 dt\left(s\left(z_k\lambda_t\frac{\delta\mathcal{I}}{\delta w_k}\right) - w_k\lambda_t\frac{\delta\mathcal{I}}{\delta w_k} (-1)^{\epsilon(z_k)} z_k[s,\lambda_t]\frac{\delta\mathcal{I}}{\delta w_k}\right). \tag{3.17}$$

Substituting in eq. (3.16) yields

$$\int_0^1 dt\left(z_k\lambda_t\frac{\delta\mathcal{I}}{\delta z_k} + w_k\lambda_t\frac{\delta\mathcal{I}}{\delta w_k}\right) = \int_0^1 dt\Bigg(s\left(z_k\lambda_t\frac{\delta\mathcal{I}}{\delta w_k}\right) - $$
$$- z_k\left((-1)^{\epsilon(z_k)}[s,\lambda_t]\frac{\delta\mathcal{I}}{\delta w_k} + \lambda_t\frac{\delta g_i}{\delta w_k}\frac{\delta\mathcal{I}}{\delta\varphi_i}\right)\Bigg). \tag{3.18}$$

The l.h.s. in eq. (3.18) gives $\mathcal{I}[z,w,\varphi] - \mathcal{I}[0,0,\varphi]$, so that

$$\mathcal{I}[z,w,\varphi] - \mathcal{I}[0,0,\varphi] = \int_0^1 dt\left[s\left(z_k\lambda_t\frac{\delta\mathcal{I}}{\delta w_k}\right) - z_k\left((-1)^{\epsilon(z_k)}[s,\lambda_t]\frac{\delta\mathcal{I}}{\delta w_k} + \lambda_t\frac{\delta g_i}{\delta w_k}\frac{\delta\mathcal{I}}{\delta\varphi_i}\right)\right]. \tag{3.19}$$

Notice that

$$[s,\lambda_t] = \left[g_i\frac{\delta(\cdot)}{\delta\varphi_i},\lambda_t\right]. \tag{3.20}$$

Let us define now

$$\mathcal{I}_1 = -\int_0^1 dt\, z_k\left((-1)^{\epsilon(z_k)}[s,\lambda_t]\frac{\delta\mathcal{I}}{\delta w_k} + \lambda_t\frac{\delta g_i}{\delta w_k}\frac{\delta\mathcal{I}}{\delta\varphi_i}\right). \tag{3.21}$$

By using $s\mathcal{I} = 0$, $s\mathcal{I}[0,0,\varphi] = 0$ and the fact that $s^2 = 0$ we get from eq. (3.19)

$$s\mathcal{I}_1 = 0\,. \tag{3.22}$$

Moreover we see from eq. (3.21) that $\mathcal{I}_1$ satisfies the condition

$$\mathcal{I}_1[z,w,\varphi]|_{z=w=0} = 0\,.$$

Thus in particular

$$s\mathcal{I}_1[0,0,\varphi] = 0\,. \tag{3.23}$$

Hence we can repeat the argument used for $\mathcal{I}$ and write $\mathcal{I}_1$ in the form

$$\mathcal{I}_1[z,w,\varphi] = \mathcal{I}_1[z,w,\varphi] - \mathcal{I}_1[0,0,\varphi]$$
$$= \int_0^1 dt\left[s\left(z_k\lambda_t\frac{\delta\mathcal{I}_1}{\delta w_k}\right) - z_k\left((-1)^{\epsilon(z_k)}[s,\lambda_t]\frac{\delta\mathcal{I}_1}{\delta w_k} + \lambda_t\frac{\delta g_i}{\delta w_k}\frac{\delta\mathcal{I}_1}{\delta\varphi_i}\right)\right]. \tag{3.24}$$

By taking into account eq. (3.20) we see that the second term in the r.h.s. of eq. (3.24) contains at least two z's.

The construction can be iterated. Assume that $\mathcal{I}_n$, $n \geq 1$ contains the product of at least n z's and assume that $s\mathcal{I}_n = 0$. Moreover, we assume that $\mathcal{I}_n[z,w,\varphi]|_{z=w=0} = 0$, which implies in particular $s\mathcal{I}_n[0,0,\varphi] = 0$. Then we can write

$$\begin{aligned}\mathcal{I}_n[z,w,\varphi] &= \mathcal{I}_n[z,w,\varphi] - \mathcal{I}_n[0,0,\varphi] \\ &= s\left[\int_0^1 dt\,\left(z_k\lambda_t\frac{\delta\mathcal{I}_n}{\delta w_k}\right)\right] + \mathcal{I}_{n+1}\,.\end{aligned} \tag{3.25}$$

In the previous equation we have defined

$$\mathcal{I}_{n+1} = -\int_0^1 dt\, z_k\left((-1)^{\epsilon(z_k)}[s,\lambda_t]\frac{\delta\mathcal{I}_n}{\delta w_k} + \lambda_t\frac{\delta g_i}{\delta w_k}\frac{\delta\mathcal{I}_n}{\delta\varphi_i}\right). \tag{3.26}$$

We now notice that $\mathcal{I}_{n+1}$ contains the product of at least $(n+1)$ z's. Moreover $s\mathcal{I}_{n+1} = 0$ and $\mathcal{I}_{n+1}(z,w,\varphi)|_{z=w=0} = 0$.

Thus we obtain that $\mathcal{I}$ is s-exact up to a term containing $(n+1)$ z's.

By the previous arguments we can therefore conclude that $\mathcal{I}[z,w,\varphi] - \mathcal{I}[0,0,\varphi]$ is s-exact as a integrated local formal power series in z. This ends the proof.

The above argument provides an explicit although recursive representation of the local integrated formal power series $\mathcal{G}$ in eq. (3.12):

$$\mathcal{G} = \int_0^1 dt\, s\left[z_k\lambda_t\frac{\delta}{\delta w_k}\left(\sum_{j=0}^{\infty}\mathcal{I}_j\right)\right], \tag{3.27}$$

where we have set $\mathcal{I}_0 \equiv \mathcal{I}$.

Of course $\mathcal{G}$ is not unique, since for any s-invariant local formal power series $\mathcal{F}$ we have that

$$\mathcal{G}' \equiv \mathcal{G} + \mathcal{F} \tag{3.28}$$

also satisfies eq. (3.12) if $\mathcal{G}$ does.

We notice that in the general case and without additional assumptions $\mathcal{G}$ in eq. (3.27) is an actual formal power series, which does not reduce to a polynomial in $\{z,w,\varphi\}$ and their derivatives.

In particular, if $\mathcal{I}[0,0,\varphi] = 0$ the previous lemma implies that the whole s-invariant $\mathcal{I}[z,w,\varphi]$ is s-exact: any s-closed local integrated formal power series vanishing at $z = w = 0$ is s-exact.

We remark that the condition

$$s\mathcal{I}[0,0,\varphi] = 0 \tag{3.29}$$

is also necessary if the (z,w)-dependence of the s-invariant $\mathcal{I}[z,w,\varphi]$ has to be cohomologically trivial. This is true independently of the coupled or decoupled nature of the doublets under investigation. Indeed assume that there exists an integrated local formal power series $\mathcal{G}[z,w,\varphi]$ such that

$$\mathcal{I}[z,w,\varphi] - \mathcal{I}[0,0,\varphi] = s\mathcal{G}[z,w,\varphi]\,. \tag{3.30}$$

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Then by the nilpotency of s and the s-invariance of $\mathcal{I}[z,w,\varphi]$ we conclude that

$$s\mathcal{I}[0,0,\varphi]=0\,. \tag{3.31}$$

Therefore eq. (3.31) is both a necessary and sufficient (by virtue of lemma 1) condition in order that the dependence of the s-invariant $\mathcal{I}[z,w,\varphi]$ on the doublets (z,w) is cohomologically trivial in the sense of eq. (1.3).

Notice that in the case of decoupled doublets eq. (3.31) is actually a consequence of the fact that $\mathcal{I}[z,w,\varphi]$ is s-invariant, i.e. of the equation

$$s\mathcal{I}[z,w,\varphi]=0 \tag{3.32}$$

once one takes the zero-th order (with respect to the grading induced by $\mathcal{N}$) component of eq. (3.32).

Lemma 1 turns out to be a useful tool in many practical applications, like for instance in the discussion of the on-shell case of the Equivalence Theorem [14].

By using lemma 1 we will now prove in theorem 1 the main result that the cohomology of s is independent of the (generally coupled) doublets (z,w). This in turn allows to characterize the full cohomology of s in terms of the cohomology of the doublets-independent component of s.

Theorem 1 *The cohomology of the nilpotent differential s in the space Σ of integrated local formal power series in $\{\varphi,z,w\}$ and their derivatives is isomorphic to the cohomology of $\bar{s}^{(0)}$ in the space $\Sigma^{(0)}$ of integrated local formal power series which only depend on φ and their derivatives:*

$$H(s,\Sigma)\approx H(\bar{s}^{(0)},\Sigma^{(0)})\,. \tag{3.33}$$

Proof. We will explicitly construct an isomorphism Φ between $H(s,\Sigma)$ and $H(\bar{s}^{(0)},\Sigma^{(0)})$. Let $\mathcal{I}\in\Sigma$ be such that

$$s\mathcal{I}=0\,. \tag{3.34}$$

We decompose $\mathcal{I}$ according to the degree induced by $\mathcal{N}$, i.e.

$$\mathcal{I}=\sum_{k=0}^{\infty}\mathcal{I}^{(k)} \tag{3.35}$$

where $\mathcal{I}^{(k)}$ belongs to the eigenspace of eigenvalue k of the counting operator $\mathcal{N}$:

$$\mathcal{N}\mathcal{I}^{(k)}=k\mathcal{I}^{(k)}\,. \tag{3.36}$$

We notice that

$$\bar{s}^{(0)}\mathcal{I}^{(0)}=0 \tag{3.37}$$

i.e. $\mathcal{I}^{(0)}$ belongs to the cohomology of $\bar{s}^{(0)}$ in $\Sigma^{(0)}$. This follows from eq. (3.34) once we consider its zero-th order in the expansion based on the grading induced by $\mathcal{N}$:

$$\left(s^{(0)}+s^{(1)}+\cdots\right)\left(\mathcal{I}^{(0)}+\mathcal{I}^{(1)}+\cdots\right)=0\Rightarrow s^{(0)}\mathcal{I}^{(0)}=0\,. \tag{3.38}$$

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The dots stand for term of higher order with respect to the grading induced by $\mathcal{N}$. Since $\mathcal{I}^{(0)}$ is independent of (z, w) the r.h.s. of eq. (3.38) entails eq. (3.37).

Let us now set

$$\Phi[\mathcal{I}] = [\mathcal{I}^{(0)}] \,, \tag{3.39}$$

where $[\mathcal{I}]$ stands for the cohomology class of $\mathcal{I}$ in $H(s, \Sigma)$ and $[\mathcal{I}^{(0)}]$ stands for the cohomology class of $\mathcal{I}^{(0)}$ in $H(\bar{s}^{(0)}, \Sigma^{(0)})$.

The map in eq. (3.39) is well-defined in cohomology since if

$$\mathcal{I} = s\mathcal{G} \tag{3.40}$$

then also

$$\mathcal{I}^{(0)} = \bar{s}^{(0)}\mathcal{G}^{(0)} \,. \tag{3.41}$$

This follows by expanding eq. (3.40) according to the grading induced by $\mathcal{N}$ once we look at the zero-th order terms:

$$\begin{aligned} &\mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \cdots = (s^{(0)} + s^{(1)} + \cdots)(\mathcal{G}^{(0)} + \mathcal{G}^{(1)} + \cdots) \\ &\quad \Rightarrow \mathcal{I}^{(0)} = s^{(0)}\mathcal{G}^{(0)} \,. \end{aligned} \tag{3.42}$$

We recover eq. (3.41) from the above equation since $\mathcal{G}^{(0)}$ is independent of (z, w).

We now show that Φ is an isomorphism.

Let us first prove that Φ is surjective. Given any $\bar{s}^{(0)}$-invariant $\mathcal{I}^{(0)}$ we show that it can be completed to a s-invariant $\mathcal{I}$. So we look for the coefficients $\mathcal{I}^{(1)}, \mathcal{I}^{(2)}, \ldots$ in such a way that

$$\mathcal{I} \equiv \mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \cdots \tag{3.43}$$

fulfills

$$s\mathcal{I} = \left(s^{(0)} + s^{(1)} + s^{(2)} + \cdots\right)\left(\mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \cdots\right) = 0 \,. \tag{3.44}$$

The proof is a recursive one. At order zero eq. (3.44) is true since $\mathcal{I}^{(0)}$ is a $\bar{s}^{(0)}$-invariant and $\mathcal{I}^{(0)}$ is (z, w)-independent, so that

$$0 = \bar{s}^{(0)}\mathcal{I}^{(0)} = s^{(0)}\mathcal{I}^{(0)} \,. \tag{3.45}$$

We next move to the first order. From the nilpotency of s we get

$$s^{(1)}s^{(0)} + s^{(0)}s^{(1)} = 0 \,. \tag{3.46}$$

Since $s^{(0)}\mathcal{I}^{(0)} = 0$ we get from the above equation

$$s^{(0)}s^{(1)}\mathcal{I}^{(0)} = 0 \,. \tag{3.47}$$

$s^{(0)}$ is a nilpotent differential with respect to which (z, w) form a set of decoupled doublets. By using the standard results on the independence of the cohomology of decoupled doublets reported in section 2 we conclude from eq. (3.47) that there must exist an integrated local formal power series $\mathcal{I}^{(1)}$ such that

$$s^{(1)}\mathcal{I}^{(0)} = s^{(0)}(-\mathcal{I}^{(1)}) \,. \tag{3.48}$$

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This in turn implies that eq. (3.44) is fulfilled at order one since from eq. (3.48) we have

$$s^{(1)}\mathcal{I}^{(0)} + s^{(0)}\mathcal{I}^{(1)} = 0\,. \tag{3.49}$$

The construction can be iterated to all orders. Assume that $\mathcal{I}$ has been constructed up to order $n-1$ by assigning the coefficients $\mathcal{I}^{(0)}, \mathcal{I}^{(1)}, \ldots, \mathcal{I}^{(n-1)}$, fulfilling

$$\sum_{j=0}^{m} s^{(j)}\mathcal{I}^{(m-j)} = 0\,, \qquad m = 0, 1, \ldots, n-1\,. \tag{3.50}$$

Then by eq. (3.50) we see that

$$s\sum_{k=0}^{n-1}\mathcal{I}^{(k)}$$

starts with a coefficient of order n, let us call it $\Delta^{(n)}$:

$$s\sum_{k=0}^{n-1}\mathcal{I}^{(k)} = \Delta^{(n)} + \cdots \tag{3.51}$$

One has explicitly

$$\Delta^{(n)} = \sum_{j=1}^{n} s^{(j)}\mathcal{I}^{(n-j)}\,. \tag{3.52}$$

Since s is nilpotent we get from eq. (3.51)

$$s^2\sum_{k=0}^{n-1}\mathcal{I}^{(k)} = s(\Delta^{(n)} + \cdots) = 0\,, \tag{3.53}$$

so that

$$(s^{(0)} + s^{(1)} + \cdots)(\Delta^{(n)} + \cdots) = 0\,. \tag{3.54}$$

We look at the lowest order contribution to the above equation, i.e. at order n, and get

$$s^{(0)}\Delta^{(n)} = 0\,. \tag{3.55}$$

By using again the results on decoupled doublets of section 2 we can show that there exists an integrated local formal power series $\mathcal{I}^{(n)}$ such that

$$\Delta^{(n)} = -s^{(0)}\mathcal{I}^{(n)}\,. \tag{3.56}$$

By taking into account eq. (3.52) we get from eq. (3.56)

$$s^{(0)}\mathcal{I}^{(n)} + \sum_{j=1}^{n} s^{(j)}\mathcal{I}^{(n-j)} = \sum_{j=0}^{n} s^{(j)}\mathcal{I}^{(n-j)} = 0\,, \tag{3.57}$$

so that eq. (3.44) is fulfilled at order n.

We now show that Φ is also one-to-one by proving that

$$\ker\Phi = \{[0]\}\,. \tag{3.58}$$

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For that purpose let us take a s-invariant $\mathcal{I}$ such that

$$\Phi([\mathcal{I}]) = [0]\,. \tag{3.59}$$

This means that the zero-th order component $\mathcal{I}^{(0)}$ of $\mathcal{I}$ fulfills

$$\mathcal{I}^{(0)} = \bar{s}^{(0)}\mathcal{G}^{(0)} \tag{3.60}$$

for some integrated local formal power series $\mathcal{G}^{(0)}$ independent of (z,w). Then

$$\begin{aligned} \mathcal{I} - s\mathcal{G}^{(0)} &= \left(\mathcal{I}^{(0)} - \bar{s}^{(0)}\mathcal{G}^{(0)}\right) + \left(\mathcal{I}^{(1)} - s^{(1)}\mathcal{G}^{(0)}\right) + \left(\mathcal{I}^{(2)} - s^{(2)}\mathcal{G}^{(0)}\right) + \cdots \\ &= \left(\mathcal{I}^{(1)} - s^{(1)}\mathcal{G}^{(0)}\right) + \left(\mathcal{I}^{(2)} - s^{(2)}\mathcal{G}^{(0)}\right) + \cdots \end{aligned} \tag{3.61}$$

In the second line of the above equation we have used eq. (3.60).

From eq. (3.61) we see that $(\mathcal{I} - s\mathcal{G}^{(0)})|_{z=w=0} = 0$. Moreover

$$\mathcal{I} - s\mathcal{G}^{(0)}$$

is s-invariant since $s\mathcal{I} = 0$ and $s^2 = 0$. Thus $\mathcal{I} - s\mathcal{G}^{(0)}$ fulfills the assumptions of lemma 1 and we conclude that

$$\mathcal{I} - s\mathcal{G}^{(0)} = s\mathcal{H} \tag{3.62}$$

for some integrated local formal power series $\mathcal{H}$. Hence we see that

$$\mathcal{I} = s(\mathcal{G}^{(0)} + \mathcal{H})\,. \tag{3.63}$$

From eq. (3.63) we conclude that $\mathcal{I}$ is s-exact. Therefore eq. (3.58) is verified and the Theorem is proven.

We notice that the injectivity of Φ can also be derived in an alternative fashion by using the fact that the cohomology of $s^{(0)}$ is concentrated in degree zero. The proof is a recursive one. By eq. (3.60) we get

$$\begin{aligned} \mathcal{I} &= \mathcal{I}^{(0)} + \mathcal{I} - \mathcal{I}^{(0)} = \bar{s}^{(0)}\mathcal{G}^{(0)} + \mathcal{I} - \mathcal{I}^{(0)} \\ &= s\mathcal{G}^{(0)} - (s - \bar{s}^{(0)})\mathcal{G}^{(0)} + \mathcal{I} - \mathcal{I}^{(0)} \\ &= s\mathcal{G}^{(0)} + \mathcal{A}_1\,, \end{aligned} \tag{3.64}$$

where $\mathcal{A}_1 \equiv -(s - \bar{s}^{(0)})\mathcal{G}^{(0)} + \mathcal{I} - \mathcal{I}^{(0)}$ is at least of order 1 in the degree induced by $\mathcal{N}$. Therefore $\mathcal{I}$ is s-exact up to order 1.

Let us now assume that $\mathcal{I}$ has been shown to be s-exact up to order k:

$$\mathcal{I} = s\mathcal{F}_{j-1} + \mathcal{A}_j\,, \qquad j = 1, 2\ldots, k \tag{3.65}$$

where $\mathcal{A}_j$ is at least of order j and can therefore be decomposed as

$$\mathcal{A}_j = \mathcal{A}_j^{(j)} + \mathcal{A}_j^{(j+1)} + \cdots \tag{3.66}$$

In the above equation $\mathcal{A}_j^{(m)}$ is the component of order m of $\mathcal{A}_j$.

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For $j = 1$ we get $\mathcal{F}_0 = \mathcal{G}^{(0)}$, by comparison of eq. (3.64) and eq. (3.65).

We now prove that eq. (3.65) is also verified at order $k+1$ if it is fulfilled up to order k. For that purpose we compute the s-variation of both sides of eq. (3.65). Since $s^2 = 0$ we get at order k:

$$s\mathcal{I} = s^2\mathcal{F}_{k-1} + s\mathcal{A}_k = s\mathcal{A}_k \,. \tag{3.67}$$

Since $s\mathcal{I} = 0$ the above equation gives

$$0 = s\mathcal{A}_k \,. \tag{3.68}$$

Let us expand eq. (3.68) according to the degree induced by $\mathcal{N}$. The first non-zero term is of order k and reads

$$0 = s^{(0)}\mathcal{A}_k^{(k)} \,. \tag{3.69}$$

Since the cohomology of $s^{(0)}$ is concentrated in degree zero there exists an integrated local formal power series $\mathcal{B}_k^{(k)}$ such that

$$\mathcal{A}_k^{(k)} = s^{(0)}\mathcal{B}_k^{(k)} \,. \tag{3.70}$$

We insert eq. (3.70) into eq. (3.65) with $j = k$ and get

$$\begin{aligned}\mathcal{I} &= s\mathcal{F}_{k-1} + \mathcal{A}_k^{(k)} + \mathcal{A}_k^{(k+1)} + \mathcal{A}_k^{(k+2)} + \cdots \\ &= s\mathcal{F}_{k-1} + s^{(0)}\mathcal{B}_k^{(k)} + \mathcal{A}_k^{(k+1)} + \mathcal{A}_k^{(k+2)} + \cdots \\ &= s(\mathcal{F}_{k-1} + \mathcal{B}_k^{(k)}) - (s - s^{(0)})\mathcal{B}_k^{(k)} + \mathcal{A}_k - \mathcal{A}_k^{(k)} \,.\end{aligned} \tag{3.71}$$

The integrated local formal power series $\mathcal{A}_{k+1}$ given by

$$\mathcal{A}_{k+1} \equiv -(s - s^{(0)})\mathcal{B}_k^{(k)} + \mathcal{A}_k - \mathcal{A}_k^{(k)} \tag{3.72}$$

is at least of order $k+1$. If we choose

$$\mathcal{F}_k \equiv \mathcal{F}_{k-1} + \mathcal{B}_k^{(k)} \tag{3.73}$$

eq. (3.71) can be rewritten as

$$\mathcal{I} = s\mathcal{F}_k + \mathcal{A}_{k+1} \,. \tag{3.74}$$

Thus eq. (3.65) is verified at order $k+1$. This concludes the recursive proof of the injectivity of Φ.

4. Conclusions

In the present paper we have discussed on general grounds the dependence of nilpotent differentials on doublets, both in the decoupled and the coupled case.

We have explicitly constructed an isomorphism between the cohomology of s in the space of integrated local formal power series depending on the set of doublets (z, w) and on φ and their derivatives and the cohomology of $\bar{s}^{(0)}$ in the space of integrated local formal power series only depending on φ and their derivatives.

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To this extent the cohomology of any nilpotent differential s in the space of integrated local formal power series is independent of doublets both in the decoupled and the coupled case.

As a final point we remark that the whole analysis was purely algebraic. No use was made of power-counting arguments.

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